Quantum Channels (part 3)

Big Picture summary / recap

	State	Evolution
Probabilities fundamental	**Ensembles** $\rho_A = \sum_i p_i \lvert\phi_i\rangle\langle\phi_i\rvert = \sum_k \lambda_k \lvert\lambda_k\rangle\langle\lambda_k\rvert$ with $\sqrt{p_i}\lvert\phi_i\rangle = \sum_k u_{ik}\sqrt{\lambda_k}\lvert\lambda_k\rangle$ ($\uparrow$ isometry)	**Kraus representation** $\mathcal{E} = \sum_i A_i (\,) A_i^\dagger \quad \sum_i A_i^\dagger A_i = I$ ($\uparrow$ obtained from diagonalizing $J(\mathcal{E})$) more generally $\Downarrow$ $\mathcal{E} = \sum_i B_i (\cdot) B_i^\dagger$ s.t. $B_i = \sum_k u_{ik} A_k$ (unitary/isometry $\nearrow$)
Universe is Unitary / "Church of the Larger Hilbert space"	**Purifications** $\lvert\psi_{AB}\rangle = \sum_k \sqrt{\lambda_k}\lvert\lambda_k\rangle \otimes \lvert k\rangle$ $\Downarrow$ or more generally $\lvert\Phi_{AC}\rangle = I_A \otimes V \lvert\Psi_{AB}\rangle$ ($\nearrow$ isometry) (captures the fact that it doesn't matter what the environment is doing	**Stinespring Dilation** $\mathcal{E}(\,) = \mathrm{Tr}_E\left(U\left((\,)\otimes\lvert 0\rangle\langle 0\rvert\right)U^\dagger\right)$ $U\lvert\psi\rangle\lvert 0\rangle = \sum_i A_i \lvert\psi\rangle\lvert i\rangle \quad \forall\psi$

The representation theorem

Actually glossed over a very important detail at the start...

We started with an operational definition of channels

That is we said that they had to satisfy the following requirements:

1) linearity: $\mathcal{E}(p\rho + (1-p)\sigma) = p\mathcal{E}(\rho) + (1-p)\mathcal{E}(\sigma)$

2) Output state needs to be a real state

ie. i. $\mathrm{Tr}(\mathcal{E}(\rho)) = 1$ (normalized)

ii. $\mathcal{E}(\rho) = \mathcal{E}(\rho)^\dagger$ (Hermitian)

iii. $\underbrace{\mathrm{eigs}(\mathcal{E}(\rho)) \geq 0}$ (positivity/non-negativity)

if this is true the operation $\mathcal{E}$ is said to be positive

In fact this isn't quite enough....

For the map to be physical we also need it to be possible to apply the map to only part of a state and still get a genuine state out

ie. one subsystem

ie. need $\underbrace{\mathcal{E}_A \otimes I_B}(\rho_{AB})$ to also be a valid quantum state

read: apply $\mathcal{E}$ to A
do nothing to B

If $\mathrm{eigs}(\mathcal{E} \otimes I(\rho)) \geq 0 \quad \forall \rho$

$\mathcal{E}$ is said to be completely positive

Positivity versus Complete Positivity

Classic example of an operation that is +ve but not completely +ve

= transpose

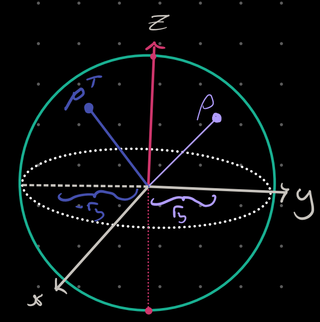
- If $\text{eigs}(\rho) \geq 0$ $\quad \text{eigs}(\rho^T) \geq 0$

This is easy to see for a single qubit

$$\rho = \tfrac{1}{2}\left(I + \underline{r} \cdot \underline{\sigma} \right)$$

$$\rho^T = \tfrac{1}{2}\left(I + \underline{r} \cdot \underline{\sigma}^T \right) \qquad \begin{pmatrix} \sigma_x \\ -\sigma_y \\ \sigma_z \end{pmatrix}$$

$$= \tfrac{1}{2}\left(I + \underline{\tilde{r}} \cdot \sigma \right) \quad \text{with } \underline{\tilde{r}} = \begin{pmatrix} r_x \\ -r_y \\ r_z \end{pmatrix}$$



This is still a valid state on the Bloch sphere

But the eigenvalues of $\mathcal{E}_T \otimes I(\rho)$ need not be +ve

e.g. $\mathcal{E}_T \otimes I\left(|\phi^+\rangle\langle\phi^+| \right) \equiv |\phi^+\rangle\langle\phi^+|^{T_A}$ known as the 'partial transpose'

$$= \mathcal{E}_T \otimes I\left(\tfrac{1}{2}\left(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|\right)\right)$$

$|0\rangle\langle 0| \to |0\rangle\langle 0|$
$|0\rangle\langle 1| \to |1\rangle\langle 0|$
$|1\rangle\langle 0| \to |0\rangle\langle 1|$
$|1\rangle\langle 1| \to |1\rangle\langle 1|$

$$= \tfrac{1}{2}\left(|00\rangle\langle 00| + |10\rangle\langle 01| + |01\rangle\langle 10| + |11\rangle\langle 11| \right)$$

$$= \tfrac{1}{2}\left(|\phi^+\rangle\langle\phi^+| + |\phi^-\rangle\langle\phi^-| + |\psi^+\rangle\langle\psi^+| - |\psi^-\rangle\langle\psi^-| \right)$$

–ve!

Therefore $\quad \text{eigs}\left(|\phi^+\rangle\langle\phi^+|^{T_A} \right) = \left\{ \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2} \right\}$

$\therefore$ $|\phi^+\rangle\langle\phi^+|^{T_A}$ is not positive

$\therefore$ $\mathcal{E}(X) = X^T$ is positive but not completely positive.

Representer Theorem

A channel $\mathcal{E}$ is:

i. Linear $\quad \mathcal{E}(\alpha\rho+\beta\sigma) = \alpha\mathcal{E}(\rho)+\beta\mathcal{E}(\sigma)$

ii. Completely positive $\quad (\mathcal{E}\otimes I)(\rho) \geq 0$

iii. trace preserving $\quad Tr(\mathcal{E}\otimes I\rho) = 1$

if & only if $\quad \mathcal{E}(\rho) = \sum_i A_i(\rho)A_i^\dagger$ with $\sum_i A_i^\dagger A_i = I$

If is easy: i. ✓

$\sum \lambda|\lambda\rangle\langle\lambda|$

ii. $\langle\psi|\sum_i A_i\otimes I(\rho)A_i^\dagger\otimes I|\psi\rangle$

$= \sum_{i\lambda}\lambda|\langle\psi|A_i\otimes I|\lambda\rangle|^2 \geq 0$

iii. $Tr(\mathcal{E}\otimes I\rho) = \sum_i Tr((A_i\otimes I)\rho(A_i^\dagger\otimes I))$

$= \sum_i Tr(A_i^\dagger A_i\otimes I\rho) = Tr(\rho)$

Only if - harder! But can use Choi representation as a tool

ie. assuming complete +ve

Let $\sigma = (I_R\otimes\mathcal{E}_Q)(\sum_{ij}|ii\rangle\langle jj|_{RQ})$

As σ is a quantum state we can write $\sigma = \sum_u \lambda_u|\lambda_u\rangle\langle\lambda_u|$

Define a map $A_u|\psi\rangle_Q = \sqrt{\lambda_u}\langle\psi^*|_R|\lambda_u\rangle_{RQ}$ +ve!

Note that $\sum_u A_u|\psi\rangle\langle\psi|_Q A_u^\dagger = \sum_u\lambda_u\langle\psi^*|_R|\lambda_u\rangle\langle\lambda_u|_{RQ}|\psi^*\rangle_R$

$= \langle\psi^*|_R\,\sigma\,|\psi^*\rangle_R$

$= \sum_{ij i'j'} \psi_{i'}^*\psi_{j'}\langle i'|_R\,|i\rangle\langle j|\otimes\mathcal{E}(|i\rangle\langle j|)\,|j'\rangle_R$

$$= \sum_{ij} \psi_i^* \psi_j \; \mathcal{E}(|i\rangle\langle j|)$$

$$= \mathcal{E}(|\psi\rangle\langle\psi|)$$

That is we have found a set of Kraus operators such that

$$\sum_k A_k (\;) A_k^\dagger = \mathcal{E}(\;)$$

$$Tr(\mathcal{E}(\rho)) = 1 \quad \forall \rho \quad \Rightarrow \quad Tr(\sum_k A_k \rho A_k^\dagger)$$

$$= Tr(\sum_k A_k^\dagger A_k \rho) = 1 \quad \forall \rho$$

$$\Downarrow \quad \sum_k A_k^\dagger A_k = I$$